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LETTER TO THE EDITOR

Lattice derivation of modular invariant partition functions on the torus

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Abstract. We express the partition function of the *ADE* lattice models on the torus in terms of partition functions of six-vertex models and which can be computed in the continuum limit. Thus we recover the partition function of minimal models with $C<1$ and new ones with $C=1$.

Since the observation [1] that modular invariant partition functions on the torus lead to the whole operator content of a conformal invariant theory [2], considerable work has been devoted [3] to deducing these partition functions from the constraints of positivity and modular invariance. Here, we obtain them from (*ADE*) lattice models. The method is to use a Temperley–Lieb equivalence [4, 5] to re-express the partition function of a given model in terms of partition functions of simple models (SOS [6]) which can be computed in the continuum limit.

The models considered below are characterised by a square lattice on a torus and a Dynkin or Coxeter diagram (figure 1) or its affine extension (figure 2). The square lattice rotated by 45° has N rows and M columns (figure 3). The rows are numbered from 0 to $N-1$, and the row numbered N is identified with that numbered 0. Similarly

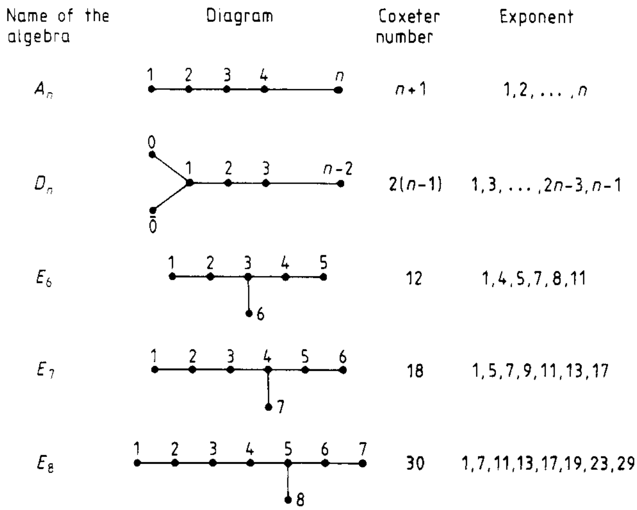

Figure 1. Coxeter diagrams.

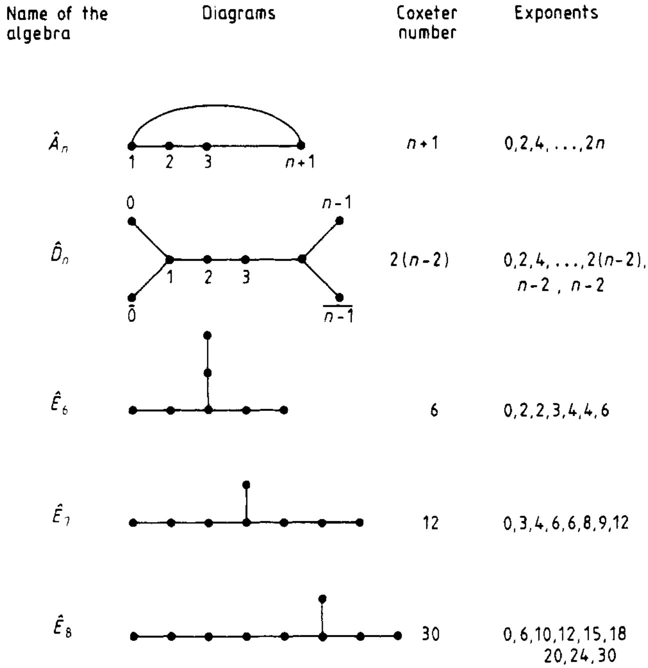

Figure 2. Coxeter diagrams.

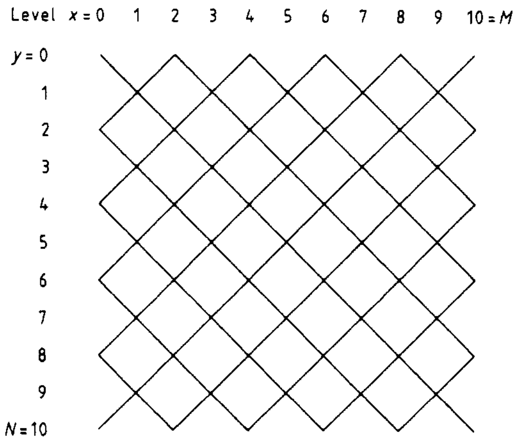

Figure 3. The lattice.

the columns are numbered from 0 to $M-1$, the column numbered M being identified with that numbered 0. Each point of the Coxeter diagram (or its affine extension) has a 'height' marked on it as in figure 1. Each site of the square lattice is assigned an arbitrary height with the only restriction that heights on any neighbouring sites are also heights on neighbouring points of the Coxeter diagram and this assignment is called a 'configuration'.

It is standard practice to express Z as the trace of a power of the 'transfer matrix' (cf [7]). We will write Z directly as this trace. The (assigned) heights $\sigma_0, \sigma_1, \ldots, \sigma_{N-1}$, $\sigma_N \equiv \sigma_0$, of a column of sites $0, 1, \ldots, N-1$ of the lattice can be used to label a basis of the Hilbert space (figure 4)

$$\mathscr{H}_N = \{|\sigma_0, \sigma_1, \ldots, \sigma_{N-1}\rangle\}. \quad (1)$$

Let us define matrices $\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_{N-1}$ by

$$\tilde{e}_i|\sigma_0, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \ldots, \sigma_{N-1}\rangle = \delta(\sigma_{i-1}, \sigma_{i+1}) \sum_{\sigma_i'} \frac{(S_{\sigma_i} S_{\sigma_i'})^{1/2}}{S_{\sigma_{i-1}}} |\sigma_0, \ldots, \sigma_{i-1}, \sigma_i', \sigma_{i+1}, \ldots, \sigma_{N-1}\rangle \quad (2)$$

where S_a is the eigenvector of the incidence matrix [8] of the Coxeter diagram corresponding to the largest eigenvalue β.

The partition function is written as

$$Z = \mathrm{Tr}(UV)^{M/2} \quad (3)$$

with

$$U = (1 + \tilde{e}_0)(1 + \tilde{e}_2) \ldots \quad (4)$$

$$V = (1 + \tilde{e}_1)(1 + \tilde{e}_3) \ldots . \quad (5)$$

We will consider one more model, hereafter called the f model, defined as follows. Heights take half-integer values and are measured modulo f. Each site of the square lattice is assigned an arbitrary height with the only restriction being that their values on the neighbouring sites differ by $+\frac{1}{2}$. The configuration space is the same as the configuration space of the $\hat{A}_{2f-1}$ model (figure 2). If $f > 1$, this model may be defined by giving a direction to each link of the square lattice instead of assigning heights to the sites, the direction arrow pointing from the smaller to the larger height. In the case $f = 1$, this is no longer possible, since one cannot decide whether $\frac{1}{2}$ is greater or less than $0 \equiv 1 \pmod 1$. In fact, if $f = 1$, then the square lattice sites are divided into two subsets, those having the height $\frac{1}{2}$ and their neighbours having the height 0; and

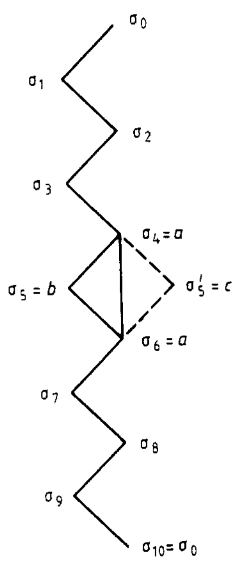

Figure 4. A column of the lattice $\tilde{e}_5$ is represented at level 5. It induces a bar between σ_4 and σ_6.

this definition by heights becomes trivial. In the case $f=1$, our model will be defined by putting arrows arbitrarily on the links of the square lattice.

For the f model, the formula corresponding to (2) is

$$\tilde{e}_i|\sigma_0,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1},\ldots,\sigma_{N-1}\rangle$$
$$=\delta(\sigma_{i-1},\sigma_{i+1})\sum_{\sigma'_i=\sigma_i\pm\frac{1}{2}} z^{\sigma_i+\sigma'_i-2\sigma_{i-1}}|\sigma_0,\ldots,\sigma_{i-1},\sigma'_i,\sigma_{i+1},\ldots,\sigma_{N-1}\rangle \qquad (6)$$

where z is a complex number. We set

$$z^{1/2}+z^{-1/2}=\beta. \qquad (7)$$

Later we shall compare an f model with a model characterised by a Coxeter diagram corresponding to the same value of β.

Expanding the trace in (3) as a sum of terms, each term can be represented graphically as follows. In the elementary square of the lattice located at the ith row and jth column ($i-j$ is odd), one puts a vertical bar if the term $\tilde{e}_j$ is picked up in the ith matrix U or V in $(UV)^{M/2}=UVUV\ldots$ (see figure 4) and one puts a horizontal bar if 1 is picked up. Thus, in each elementary square, there is either a vertical bar or a horizontal bar; the sites of the lattice are grouped into clusters; all sites of a cluster have the same height. These clusters can be separated by boundaries drawn on the dual lattice (figure 5).

One can further simplify this graphical representation (figures 6 and 7). Each cluster of sites is now represented by a point; points representing clusters with a common boundary are joined by a line. Clusters wrapping around the torus one way or the other are represented by thick points and other clusters (homotopic to a point) are represented by thin points. There is always one and only one cycle of thick points

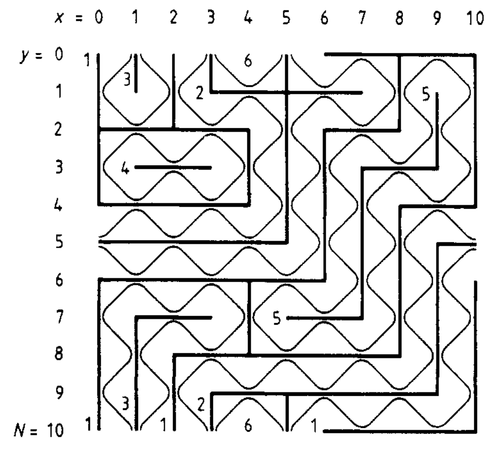

Figure 5. A cluster decomposition.

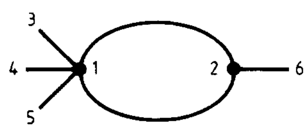

Figure 6. The graph corresponding to the cluster decomposition, figure 5.

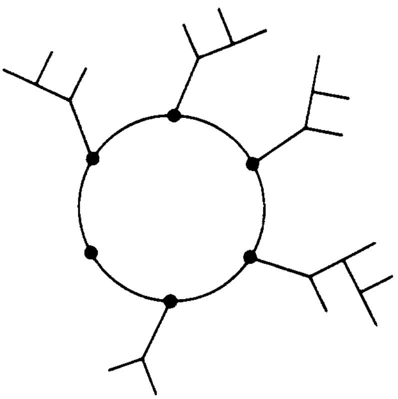

Figure 7. The graph of a cluster decomposition.

(which may consist of a single thick point) and all other points are joined in a tree structure attached to the cycle of thick points. The contribution of such a graph to the partition function is a sum of terms, one for each allowed assignment of heights at points of the graph (configuration). The contribution of a configuration is a product of factors coming for each cluster and can be obtained as follows.

Let us go back to the lattice representation of figure 5. Consider two possible types of corners (1) and (2) (figure 8) that can occur at a boundary of a cluster with height a. For corners of type (1), the height a on the left (right) of the corner corresponds to $\sigma_i(\sigma_i')$ in the expression (2) of $\tilde{e}_i$ and one picks up a factor $S_a^{1/2}$. For corners of type (2), the height a above (below) corresponds to $\sigma_{i+1}(\sigma_{i-1})$ and one picks up a factor $S_a^{-1/2}$ if one takes the product of the factor $S_a^{\pm 1/2}$ around a boundary. There are three cases. Boundaries that surround the cluster take a factor S_a. Boundaries that are surrounded by it take a factor S_a^{-1}. Boundaries that wind around the torus take a factor 1. The resulting factor is $S_a^{b_+-b_-}$ where $b_+(b_-)$ is the number of boundaries surrounding (surrounded by) the cluster. If a cluster does not wind around the torus, it is surrounded by one boundary which is represented by the link connecting the corresponding point to a cycle. The links around a cycle correspond to the clusters which wind around the torus.

We want to express Z, the partition function of a model characterised by a Coxeter diagram, as a linear combination of Z_f, the partition functions of f models,

$$Z = \sum_{f \in \mathbb{N}} a_f Z_f. \tag{8}$$

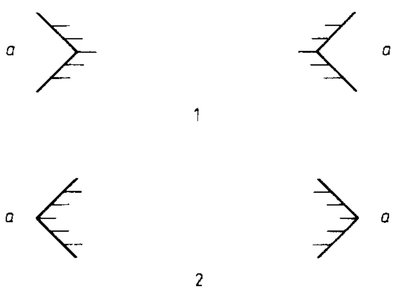

Figure 8. Two types of corner in a contour bounding the cluster a. The hatched line corresponds to the outside of a cluster.

In fact, we shall determine the coefficients a_f such that the equality is true for each term in the development of the trace (3):

$$\bar{Z} = \sum_{f \in \mathbb{N}} a_f \bar{Z}_f \tag{9}$$

where $\bar{Z}(\bar{Z}_f)$ is the contribution of an arbitrary graph to the partition function.

Given a graph, one sees that if one sums over the heights of a cluster represented by a point at the extremity of only one link ($b_+ = 1$, $b_- = 0$), $\bar{Z}(\bar{Z}_f)$ is equal to β times the contribution $\bar{Z}'(\bar{Z}'_f)$ of the graph with the link removed. Thus, to determine the a_f, we can restrict ourselves to consider cycles (see figure 9). For cycles, the contribution of a height configuration is 1. The problem is therefore reduced to find a_f such that for all even N one has

$$\mathrm{Tr}\, 1_N = \sum_{f \in \mathbb{N}} a_f \,\mathrm{Tr}\, 1^f_N \tag{10}$$

where $\mathrm{Tr}\, 1_N (\mathrm{Tr}\, 1^N_f)$ counts the number of closed paths of length N on the Coxeter $(\hat{A}_{2f-1})$ diagram. To compute $\mathrm{Tr}\, 1_N$ (the following argument is due to Kostov) let us define the matrix $G^N_{\sigma\sigma'}$ that counts the number of paths of length N going from σ to σ'. Clearly one has

$$\mathrm{Tr}\, 1_N = \mathrm{Tr}\, G^N = \mathrm{Tr}(G^1)^N \tag{11}$$

where G^1 is the incidence matrix of the diagram and its eigenvalues are $\lambda_m = 2\cos(m\pi/h)$. h is the Coxeter number and m are exponents [8] (see figures 1 and 2). For the f models (A_{2f-1} diagram) the eigenvalues of G^1 are equal to

$$\lambda_j = 2\cos\left(\frac{\pi}{2f} j\right) \qquad j = 0, 2, 4, \ldots, 4f-2.$$

If we substitute the value of the trace in (10) we get

$$\sum_{m \in \text{exponents}} \left(2\cos\frac{m\pi}{h}\right)^N = 2 \sum_{f \in N} a_f \sum_{j=0,1,\ldots,f-1} \left(2\cos\frac{\pi j}{f}\right)^N. \tag{12}$$

The equality must be true for all even $N \geqslant 0$. We therefore deduce the equation for the a_f:

$$n_m = 2 \sum_{h/f \text{ divides } m} a_f \qquad 1 \leqslant m \leqslant h \tag{13}$$

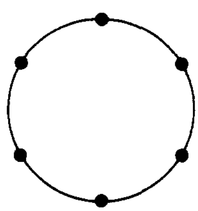

Figure 9. The cluster decomposition, figure 7, after removing the clusters homotopic to a point.

where n_m denotes the degeneracy $(0, 1, 2, \ldots)$ of the exponent m (figures 1 and 2). $a_f = 0$ if f does not divide h. The resulting partition functions are listed in appendix 1.

In the continuum limit, we expect the discrete height variable of the f model to be replaced by a continuous field φ_f defined modulo f. We assume that the partition function takes the following simple form:

$$Z_f = \int [\mathrm{D}\varphi_f] \exp\left(-\int \frac{g}{4\pi}(\nabla\varphi_f)^2\, \mathrm{d}^2x\right). \tag{14}$$

The x integration is performed on a torus defined by its two complex periods ω_1, ω_2 such that $\tau = \omega_2/\omega_1$ lies in the upper half-plane.

To evaluate (14), one first performs the integral over fields φ periodic around the torus, the result being [9]

$$Z(g) = \frac{\sqrt{g}}{\tau_1^{1/2}\eta(q)\eta(\bar{q})} \tag{15}$$

where τ_1 is the imaginary part of τ, η is the Dedekind function and

$$\begin{aligned} \eta(q) &= q^{1/24} \prod_{N=1}^{\infty} (1 - q^N) \\ q &= \exp(2\mathrm{i}\pi\tau). \end{aligned} \tag{16}$$

From the expression (15) of Z, one easily deduces the expression $Z_{m,m'}$ when the field φ takes a discontinuity $m(m')$ along $\omega_1(\omega_2)$. Z_f is obtained by summing over $Z_{m,m'}$ with m, m' a multiple of f. The computation done in (8) gives

$$Z_f(g) = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{\substack{e \in \mathbb{Z}/f \\ M \in \mathbb{Z}f}} q^{-\frac{1}{4}(e/\sqrt{g} + M\sqrt{g})^2} \bar{q}^{-\frac{1}{4}(e/\sqrt{g} - M\sqrt{g})^2}. \tag{17}$$

It was shown in [1] how to deduce from (17) the scaling dimensions and spins of operators in the theory.

The parameter g in (14) is not fixed and we assume that [10]

$$\begin{aligned} g &= (1 - 1/h) \qquad \text{for the diagrams in figure 1} \\ g &= 1 \qquad \text{for the diagrams in figure 2.} \end{aligned} \tag{18}$$

In the case of the diagrams in figure 1, another value of g: $g' = (1 + 1/h)$ can be reached if we modify the models in allowing sites not to be occupied by any height (vacancies) [12].

The partition function corresponding to the diagrams (figure 1) had been obtained from the constraints of modular invariance [3] and were rewritten in terms of 'Coulomb gas partition functions' in [9]. The partition function of $\hat{D}_4$ is the partition function of the four-state Potts model [12].

The configuration space of the $\hat{A}_{2n-1}$ model is the same as the one of the $n = f$ model and we can use the definition (6) of $\tilde{e}_i$ to define a one-parameter deformation of the transfer matrix (z is a complex number of modulus 1) such that the model remains self-dual on the hole line. After performing a similarity transformation of the transfer matrix the weights can be made positive and are given by those of the F model [6] (see appendix 1). Using the parametrisation (18) of g we obtain the partition function of the deformed models

$$Z = Z_n(g). \tag{19}$$

In the $\hat{D}_{n+2}$ case, it is possible to generalise an argument of Wegner [13] in the Ashkin–Teller case to map the configurations of the model onto configurations of an $\hat{A}_{2n-1}$ model where the heights are defined modulo a sign. From the F model weights we obtained a one-parameter family of transfer matrices (see appendix 2). The partition function is given by (14), with $f=2n-1$, but now, due to the indeterminacy of the sign of the heights, the field φ_f is identified with $-\varphi_f$. The resulting partition function was computed in [12]:

$$Z=\tfrac{1}{2}(Z_n(g)+2Z_2(1)-Z_1(1)). \tag{20}$$

In conclusion we have reduced the problem of computing the partition function of the *ADE* lattice models on the torus to a simple system of linear equations (13).

This work suggests that the comparison with f models can also be done at the level of the operators [8].

I wish to thank J Cardy, P Di Francesco, H Saleur and J B Zuber for interesting me in this problem. I would also like to thank M Mehta and I Kostov for an illuminating discussion.

Appendix 1. List of partition functions

A_n:	$\tfrac{1}{2}(Z_{n+1}-Z_1)$
D_n:	$\tfrac{1}{2}(Z_{2(n-1)}-Z_{n-1}+Z_2-Z_1)$
E_6:	$\tfrac{1}{2}(Z_{12}-Z_6-Z_4+Z_3+Z_2-Z_1)$
E_7:	$\tfrac{1}{2}(Z_{18}-Z_9-Z_6+Z_3+Z_2-Z_1)$
E_8:	$\tfrac{1}{2}(Z_{30}-Z_{15}-Z_{10}-Z_6+Z_5+Z_3+Z_2-Z_1)$
$\hat{A}_{2n-1}$:	Z_n
$\hat{D}_n$:	$\tfrac{1}{2}(Z_{n-2}+2Z_2-Z_1)$
$\hat{E}_6$:	$\tfrac{1}{2}(2Z_3+Z_2-Z_1)$
$\hat{E}_7$:	$\tfrac{1}{2}(Z_4+Z_3+Z_2-Z_1)$
$\hat{E}_8$:	$\tfrac{1}{2}(Z_5+Z_3+Z_2-Z_1)$.

Appendix 2

The one-parameter family of weights which determine a self-duality line for the $\hat{A}_{2n-1}$ and $\hat{D}_n$ models.

The heights are labelled as in figure 2.

The weights are given by

$$\begin{matrix} & \sigma_3 & \\ \sigma_1 & & \sigma_4 \\ & \sigma_2 & \end{matrix} = W(\sigma_1\sigma_2|\sigma_3\sigma_4)$$

$$= W(\sigma_4\sigma_2|\sigma_3\sigma_1)$$

$$= W(\sigma_1\sigma_3|\sigma_2\sigma_4)$$

$\hat{A}_{2n-1}$.

$$W(\sigma \pm 1, \sigma \,|\, \sigma, \sigma \mp 1) = 1$$

$$W(\sigma \pm 1, \sigma \,|\, \sigma, \sigma \pm 1) = z + z^{-1} = \omega$$

$$W(\sigma, \sigma \pm 1 \,|\, \sigma \mp 1, \sigma) = 1.$$

$\hat{D}_n$. The weights remain unchanged under the transformations:

$$\sigma \leftrightarrow n-1-\sigma \qquad 0 \leqslant \sigma \leqslant n-1 \qquad \bar{0} \leftrightarrow \overline{n-1}$$

$$0 \leftrightarrow \bar{0}$$

$$n-1 \leftrightarrow \overline{n-1}.$$

They are given by the preceding expression if all σ_i around the square are different from 0, $\bar{0}$, $n-1$, $\overline{n-1}$. The remaining weights are given by

$$W(1, 0 \,|\, 0, 1) = \omega$$

$$W(1, 2 \,|\, 0, 1) = 1$$

$$W(1, \bar{0} \,|\, 0, 1) = \omega - 1$$

$$W(0, 1 \,|\, 1, 0) = \tfrac{1}{2}(\omega + 1)$$

$$W(0, 1 \,|\, 1, 2) = 1/\sqrt{2}$$

$$W(\bar{0}, 1 \,|\, 1, 0) = \tfrac{1}{2}(\omega - 1).$$

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